

Approximation for directed percolation in $d=1+1$

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We introduce an approximation specific to a continuous variable formulation of directed percolation, which is strictly equivalent to $(1+1)$ -dimensional directed bond percolation. We find that the critical exponent associated with the order parameter (percolation probability) is $\beta=(1-1/\sqrt{5})/2=0.276\,393\,202\dots$, in remarkable agreement with the best current numerical estimate $\beta=0.276\,486(8)$.

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Directed percolation (DP) [1–4] is a useful paradigm for dynamical phase transitions between an active or spreading phase and an extinct or absorbing phase. Models in the DP class of universality are involved in the description of catalytic reactions [5], surface dynamics [6], porous systems [4], granular media [7], epidemics, calcium dynamics in cells [4], developed turbulence, and coupled maps [8], and have implications for other statistical physics models (polymers, Potts model, friendly walkers) [9]. Recently, Hinrichsen summarized the large scope of possible physical applications of DP [4], which led Grassberger to conjecture that the DP universality class should describe any continuous phase transition from a fluctuating active phase into a *single* absorbing phase, in the absence of quenched disorder and special symmetries [10]. In a sense, DP plays a similar role in the study of dynamical phase transitions as the Ising model for continuous equilibrium phase transitions [4].

Despite its ubiquity, DP is maybe the only major statistical physics model that has not yet been successfully solved in one spatial dimension (+ time), probably due to its lack of conformal invariance.

Let us recall the original model of directed bond percolation in $d=1+1$, describing the propagation of a fluid in a two-dimensional (2D) porous medium. On a square lattice tilted at 45° , a fraction of bonds p is chosen at random to be active, whereas the remaining bonds stay inactive or broken. The “fluid” starts from the top row, and propagates downward, only passing through the active bonds. One then defines the order parameter $n(p,t)$, which measures the average density of occupied sites at row t . $n(p,t)$ happens to coincide with the probability that at least one site at row t is still active (percolation probability).

In the stationary limit $t \rightarrow +\infty$, the order parameter tends to a constant value $n(p)$, which is zero below $p_c = 0.644\,700\,185(5)$ [11,12], and behaves as $n(p) \sim (p - p_c)^\beta$, near p_c . This defines the universal critical exponent $\beta=0.276\,486(8)$ [11,12]. Note that the best analytical estimate for β so far is $\beta=1/2$ [13], obtained within the independent interval approximation, which, however, predicts a first order transition. This result has been improved using the enhanced coherent anomaly numerical method, leading to $\beta \approx 0.28$, and a value of p_c 5% off (using another model in the DP class) [14].

The dynamics of bond DP can be defined more precisely. Introducing $n_i(t) = 1$ [$n_i(t) = 0$], when the i th site at row t is active or occupied (inactive or empty), $n_i(t)$ satisfies the following recursion relation:

$$n_i(t+1) = a_i n_i(t) + b_i n_{i+1}(t) - a_i b_i n_i(t) n_{i+1}(t), \quad (1)$$

where $a_i(t)$ and $b_i(t)$ are independent random variables taking the value 1 with probability p (if the corresponding ongoing bond is active), and 0 otherwise.

In relation to self-organized criticality, it has been recognized that directed bond percolation is strictly equivalent to a continuous dynamical model, but involving no external parameter (like p in DP) [15–18]. On a 1D lattice, we define the continuous variables $x_i(t)$ as satisfying the recursion relation

$$x_i(t+1) = \min\{\max[x_i(t), z_i], \max[x_{i+1}(t), z'_i]\}, \quad (2)$$

where $z_i(t)$ and $z'_i(t)$ are independent random variables uniformly distributed between 0 and 1. It can be easily shown that the n_i 's for directed bond percolation and the x_i 's are very simply related [15–18]:

$$n_i(t) = \theta(p - x_i(t)), \quad (3)$$

where $\theta(\cdot)$ is the usual Heaviside step function. For instance, if site i is active [$n_i(t) = 1$ or $x_i(t) \leq p$] and site $i+1$ is inactive [$n_{i+1}(t) = 0$ or $x_{i+1}(t) > p$], then at time $t+1$ Eq. (2) implies that site i is active with the probability p that $z_i \leq p$, which coincides with the probability that the link between site i at time t and site i at time $t+1$ is active in the usual discrete formulation of DP. Similarly, if both sites are active [$n_i(t) = n_{i+1}(t) = 1$, $x_i(t) \leq p$, $x_{i+1}(t) \leq p$], then site i is active at time $t+1$, provided $z_i \leq p$ or $z'_i \leq p$, which happens with probability $2p - p^2$, equal to the probability that at least one link is active in the usual formulation. Finally, if both sites are inactive [$n_i(t) = n_{i+1}(t) = 0$, $x_i(t) > p$, $x_{i+1}(t) > p$], then $x_i(t+1) > p$ whatever z_i and z'_i , and the resulting site is inactive, as expected.

In the large time limit, the x_i 's are distributed according to a stationary probability distribution $\rho(x)$, and

$$n(p) = \int_0^p \rho(x) dx. \quad (4)$$

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Now, averaging Eq. (1) [or equivalently Eq. (2)], we find in the stationary limit,

$$\frac{2p-1}{p^2}n(p) = \frac{2p-1}{p^2}\langle n_1 \rangle = \langle n_1 n_2 \rangle \quad (5)$$

$$= \int_0^p \int_0^p \rho_2(x_1, x_2) dx_1 dx_2, \quad (6)$$

where $\rho_2(x_1, x_2)$ is the nearest neighbor correlation function of the x_i 's. In mean field (MF) theory, one makes the approximation $\langle n_1 n_2 \rangle \approx \langle n_1 \rangle^2$, leading to

$$n_{MF}(p) = \frac{2p-1}{p^2}, \quad \rho_{MF}(p) = \frac{2(1-p)}{p^3} \quad (7)$$

for $p \geq p_{cMF} = 1/2$. From now, we study the stationary state of DP in terms of the continuous model defined by Eq. (2). We first notice that

$$x_1(t+1) = \min[x_1(t), x_2(t)] \quad \text{with probability} \quad (8)$$

$$p_{\min} = \min[x_1(t), x_2(t)],$$

$$x_1(t+1) = \max[x_1(t), x_2(t)] \quad \text{with probability} \quad (9)$$

$$p_{\max} = \max[x_1(t), x_2(t)]\{1 - \max[x_1(t), x_2(t)]\},$$

so that there is a nonzero probability that $x_1(t+1) = x_2(t+1)$ exactly. Hence, the two-point correlation function of the x_i 's, $\rho_2(x_1, x_2)$, should include a $\delta(x_1 - x_2)$ contribution [$\delta(\cdot)$ is the Dirac peak distribution]. Thus, in all generality, we write $\rho_2(x_1, x_2)$ in the following form:

$$\rho_2(x_1, x_2) = \tilde{\rho}_2(x_1, x_2) + \rho(x_1)g(x_1)\delta(x_1 - x_2), \quad (10)$$

which defines $g(p)$ as the probability that $x_2 = p$, conditional on the fact that its neighbor $x_1 = p$. We then define $f(x_1, x_2)$ through the relation

$$\tilde{\rho}_2(x_1, x_2) = \rho(\min(x_1, x_2))f(x_1, x_2), \quad (11)$$

noting that as $\rho(p)$ diverges near p_c (since $\beta < 1$), $\rho(\min(x_1, x_2)) > \rho(\max(x_1, x_2))$, at least near p_c [numerically $\rho(p)$ appears to be a strictly decreasing function, as in mean field theory]. We expect that $f(x_1, x_2)$ is a smooth function of order unity. In fact, contrary to the MF approach (where one assumes that $\langle n_1 n_2 \rangle \sim [n(p)]^2$), the correlation functions all behave as $n(p)$ near p_c . Indeed, as $p_c > 1/2$, Eq. (5) implies that $\langle n_1 n_2 \rangle \sim n(p) \sim (p - p_c)^\beta$, such that $\tilde{\rho}_2(p, p) \sim \rho(p) \sim (p - p_c)^{-(1-\beta)}$ [instead of $\rho(p)^2 \sim (p - p_c)^{-2(1-\beta)}$, predicted by MF theory]. A natural guess for $f(x_1, x_2)$ is provided by the general statement that, although MF theory is inept at describing correlation functions near p_c , it still leads to reasonably accurate amplitude ratios between them, for all values of p (at least for short range correlation functions like $\langle n_1 n_2 \rangle$). Hence, this prompts the introduction of the key approximation

$$f(x_1, x_2) \approx f_{MF}(x_1, x_2) \quad (12)$$

$$= \frac{\rho_{MF}(x_1)\rho_{MF}(x_2)}{\rho_{MF}(\min(x_1, x_2))} \quad (13)$$

$$= \rho_{MF}(\max(x_1, x_2)). \quad (14)$$

In the following, we make the more general ansatz

$$\tilde{\rho}_2(x_1, x_2) = \rho(\min(x_1, x_2))f(\max(x_1, x_2)), \quad (15)$$

where $f(p)$ is not necessarily equal to $\rho_{MF}(p)$. $\rho(p)$, $f(p)$, and $g(p)$ are not independent functions as they are related together by Eq. (5), and by the probability conservation constraint

$$\rho(p) = \int_0^1 \rho_2(x, p) dx. \quad (16)$$

From Eq. (5) and Eq. (16), and after straightforward calculations, we obtain the two relations

$$n(p) = \exp\left[-\int_p^1 \frac{2f(x) - \rho_{MF}(x)}{n_{MF}(x) - g(x)} dx\right], \quad (17)$$

$$g(p) + \int_p^1 f(x) dx + f(p) \frac{n(p)}{\rho(p)} = 1. \quad (18)$$

From Eq. (17) and Eq. (18), one can obtain a first order differential equation for $F(p) = f(p) - \rho_{MF}(p)$, involving only p and $g(p)$. This equation can be shown to have only $F=0$ as a global solution satisfying the boundary conditions and the physical constraints. Hence, and in complete accordance with the physical argument given in Eq. (14), we find

$$f(p) = \rho_{MF}(p). \quad (19)$$

Quite remarkably, for this precise form for $f(p)$, Eq. (18) is now satisfied for any choice of $g(p)$, so that we are left with Eq. (17) as the only nontrivial relation between $n(p)$ and $g(p)$.

Now, as $n(p)$ vanishes at p_c , we expect the function involved in the integral of Eq. (17) to develop a single pole at p_c , of residue β , so that $n(p) \sim (p - p_c)^\beta$, near p_c . This leads to

$$g(p_c) = n_{MF}(p_c) = \frac{2p_c - 1}{p_c^2}, \quad (20)$$

$$\beta = \left(1 - \frac{g'(p_c)}{\rho_{MF}(p_c)}\right)^{-1}. \quad (21)$$

Note that Eq. (20) is in fact an *exact identity*, which does not rely on the present approximation on $\tilde{\rho}_2(x, y)$, which is not involved in Eq. (20).

In order to achieve our goal of computing $n(p)$, we need a further relation for $g(p)$. This can be obtained by writing the exact stationary equation for $g(p)$. We first define $\langle \bullet \bullet \bullet \rangle$ as the probability of having three consecutive sites

with $x_i=p$, divided by $\rho(p)$ (that is, conditional on having one site with $x_i=p$). In the same manner, we define $\langle \bullet \bullet \bar{p} \rangle$ ($\langle \bullet \bullet p \rangle$) as the probability of having two consecutive sites with $x_1=x_2=p$, and $x_3>p$ ($x_3<p$), divided by $\rho(p)$. Finally, in the stationary limit, we obtain the exact relation

$$\begin{aligned} g(p) &= (2p-p^2)^2 \langle \bullet \bullet \bullet \rangle + 2(2p-p^2)[p \langle \bullet \bullet \bar{p} \rangle \\ &\quad + p(1-p) \langle \bullet \bullet p \rangle] + p^2 \langle \bar{p} \bullet \bar{p} \rangle + 2p^2(1-p) \langle \underline{p} \bullet \bar{p} \rangle \\ &\quad + p^2(1-p)^2 \langle \underline{p} \bullet p \rangle + p^2 \langle \bullet \bar{p} \bullet \rangle \\ &= \langle \bullet \bullet \bullet \rangle. \end{aligned} \quad (22)$$

For instance, the first term $(2p-p^2)^2 \langle \bullet \bullet \bullet \rangle$ represents the fact that a configuration $\bullet \bullet [x_1(t+1)=x_2(t+1)=p]$ at time $t+1$ can arise from a configuration $\bullet \bullet \bullet [x_1(t)=x_2(t)=x_3(t)=p]$ at time t , provided that x_1 and x_2 are preserved by the transformation of Eq. (2). This happens with probability

$$(p_{\min} + p_{\max})^2 = [p + p(1-p)]^2 = (2p-p^2)^2; \quad (23)$$

hence the coefficient in Eq. (22) [p_{\min} and p_{\max} were defined in Eq. (8) and Eq. (9)].

Equation (22) relates $g(p)$ to three-point correlation functions, and cannot be exploited unless an additional approximation is introduced. We will factor these three-point correlation functions into products of two-point correlation functions, according to the usual mean field scheme. Introducing p_+ as the probability that $x_2>p$, conditional on the fact that $x_1=p$, we obtain

$$[1-g(p)]p_+ = \frac{\int_p^1 \tilde{\rho}_2(x,p) dx}{\rho(p)} \quad (24)$$

$$= \int_p^1 \rho_{MF}(x) dx = \frac{(1-p)^2}{p^2}, \quad (25)$$

where Eq. (24) is an exact identity. We give below a few examples of three-point correlation functions computed according to this MF factorization scheme:

$$\langle \bullet \bullet \bullet \rangle = g(p)^2, \quad (26)$$

$$\langle \bullet \bullet \bar{p} \rangle = g(p)[1-g(p)]p_+ = g(p) \frac{(1-p)^2}{p^2}, \quad (27)$$

$$\langle \underline{p} \bullet \bar{p} \rangle = [1-g(p)]^2 p_+(1-p_+) \quad (28)$$

$$= \left[\frac{2p-1}{p^2} - g(p) \right] \frac{(1-p)^2}{p^2}. \quad (29)$$

Inserting the MF form for the three-point correlation functions into Eq. (22), we finally obtain a closed equation for $g(p)$,

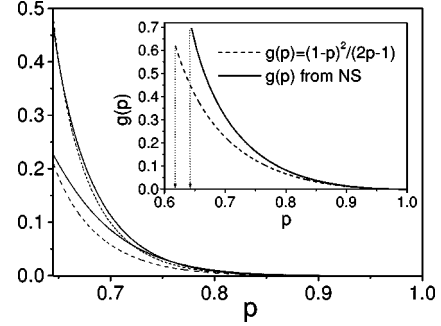


FIG. 1. We plot $\langle \bullet \bullet \bullet \rangle$ (two top lines) and $\langle \bullet \bullet \bar{p} \rangle$ as obtained from numerical simulations (full lines), and as given by Eq. (26) and Eq. (27) (dashed lines), where the numerical value of $g(p)$ has been inserted in these expressions. Note that $\langle \bullet \bullet \bullet \rangle \approx g(p)^2$, especially near p_c . These functions all vanish as $(1-p)^4$ near $p=1$, as predicted by Eq. (26) and Eq. (27). Inset: comparison between the numerical $g(p)$ and the present theory. In all figures in this paper, we have simulated a system of $N=300\,000$ sites, averaged over 100 samples. Physical quantities in the stationary state have been estimated by averaging them between $t=300\,000$ and $t=310\,000$.

$$g(p) = [1-p + pg(p)]^2 + (1-p)^2 g(p)[1-g(p)], \quad (30)$$

which can be readily solved, leading to

$$g(p) = \frac{(1-p)^2}{2p-1}. \quad (31)$$

p_c and β can now be calculated by expressing the conditions of Eq. (20) and Eq. (21). We obtain

$$p_c = g(p_c) = \tau = \frac{\sqrt{5}-1}{2} = 0.618\,033\,989\dots, \quad (32)$$

$$\beta = \frac{1}{2} \left(1 - \frac{1}{\sqrt{5}} \right) = 0.276\,393\,202\dots, \quad (33)$$

where τ is the golden mean. p_c is only in fair agreement with the best numerical estimate $p_c = 0.644\,700\,185(5)$ [11,12], although this represents a definite improvement when compared to the mean field value $p_c = 1/2$. Note that the exact identity Eq. (20) implies that getting $p_c > 1/2$ necessitates the introduction of a nontrivial function $g(p)$, which is zero in the MF. The β exponent is in remarkable agreement (relative accuracy of 0.034%) with the best available numerical value $\beta = 0.276\,486(8)$ [11,12]. However, it is not claimed that this result is by any means exact.

The fact that the relation $\langle \bullet \bullet \bullet \rangle = g(p)^2$ seems to be exactly satisfied numerically at p_c could explain this agreement, which also implies that the MF factorization of Eq. (22) is quantitatively correct near p_c [the three-point correlation functions appearing in Eq. (22) are not independent and are related to $\langle \bullet \bullet \bullet \rangle$ by various sum rules]. This is illustrated in Fig. 1, where the exact numerical $g(p) = \langle \bullet \bullet \bullet \rangle$, $\langle \bullet \bullet \bullet \rangle$, and $\langle \bullet \bullet \bar{p} \rangle$ are plotted with their theoretical counterparts. In principle, it is possible to improve

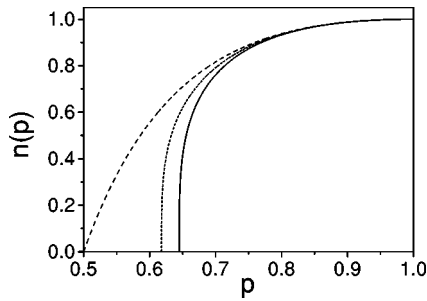


FIG. 2. We plot $n(p)$ as given by mean field theory (dashed line), the present theory [Eq. (34); dotted line], and numerical simulations (full line). Note that the three curves coincide near $p=1$, as mean field theory becomes exact in this limit. A fit of the numerical data to the functional form of Eq. (34), where τ becomes a fitting parameter, cannot be distinguished from the actual data.

this calculation by using exact relations [similar to Eq. (22)] involving higher order correlation functions ($\langle \bullet \bullet \bullet \bullet \rangle, \langle \bullet \bullet \bullet \bullet \bullet \rangle, \dots$). In practice, the calculation becomes increasingly complicated and probably intractable.

Now, $f(p)$ and $g(p)$ being known, the percolation probability can easily be computed by using Eq. (17):

$$n(p) = p^{-2} \left[\frac{(p-\tau)(2+\tau-p)}{\tau} \right]^\beta \times \left[\frac{(p-1+\tau)(p+1+\tau)}{1+\tau} \right]^{1-\beta}. \quad (34)$$

In Fig. 2, we compare this result with the numerically extrapolated stationary percolation probability, and with the MF result of Eq. (7).

Finally, in Fig. 3, in order to test the validity of our basic approximation Eq. (14), we plot $f(x_1, x_2)$ [defined in Eq. (11)] as a function of $\max(x_1, x_2)$. We find that this scatter plot is reasonably aligned around an effective curve, and that $f_{MF}(x_1, x_2) = \rho_{MF}(\max(x_1, x_2))$ appears to be a lower bound for the actual $f(x_1, x_2)$.

In conclusion, we have introduced an approximation for a

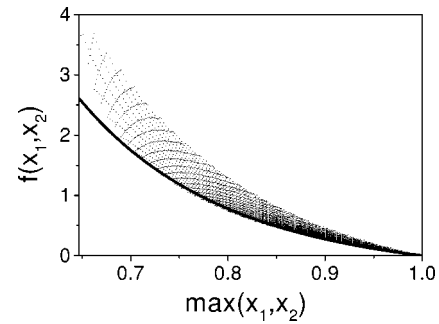


FIG. 3. We plot $f(x_1, x_2)$ defined in Eq. (11) as a function of $\max(x_1, x_2)$ (scatter plot using a 200×200 mesh for $(x_1, x_2) \in [p_c, 1]^2$). This is compared with the approximation central to this paper: $f(x_1, x_2) = \rho_{MF}(\max(x_1, x_2))$ (thick line). We observe that the theoretical expression seems to be a lower bound for the actual $f(x_1, x_2)$, and that the dispersion [due to the explicit dependence on $\min(x_1, x_2)$] is weak enough so that the scatter plot tends to align around an effective curve.

continuous model strictly equivalent to directed bond percolation. In this language, this approximation amounts to properly modeling the correlation function $\rho_2(p_1, p_2)$ relating the properties of directed bond percolation for two different percolation parameters p_1 and p_2 . By assuming that amplitude ratios are correctly described by mean field theory, we end up with a precise description of the percolation probability. In particular, we find an exponent β in remarkable agreement with the best available numerical simulations.

It would be interesting to exploit the present approach in order to describe the dynamical properties of DP. This study is currently in progress.

This approach could also prove useful in tackling the notably difficult problem of parity conserving branching annihilating walks [3]. This universality class is exemplified by the reaction-diffusion model of diffusing particles A , involving annihilation ($A+A \rightarrow \emptyset$) and branching ($A \rightarrow A+A+A$) processes. This problem has so far eluded all manner of theoretical approaches in $d=1+1$.

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